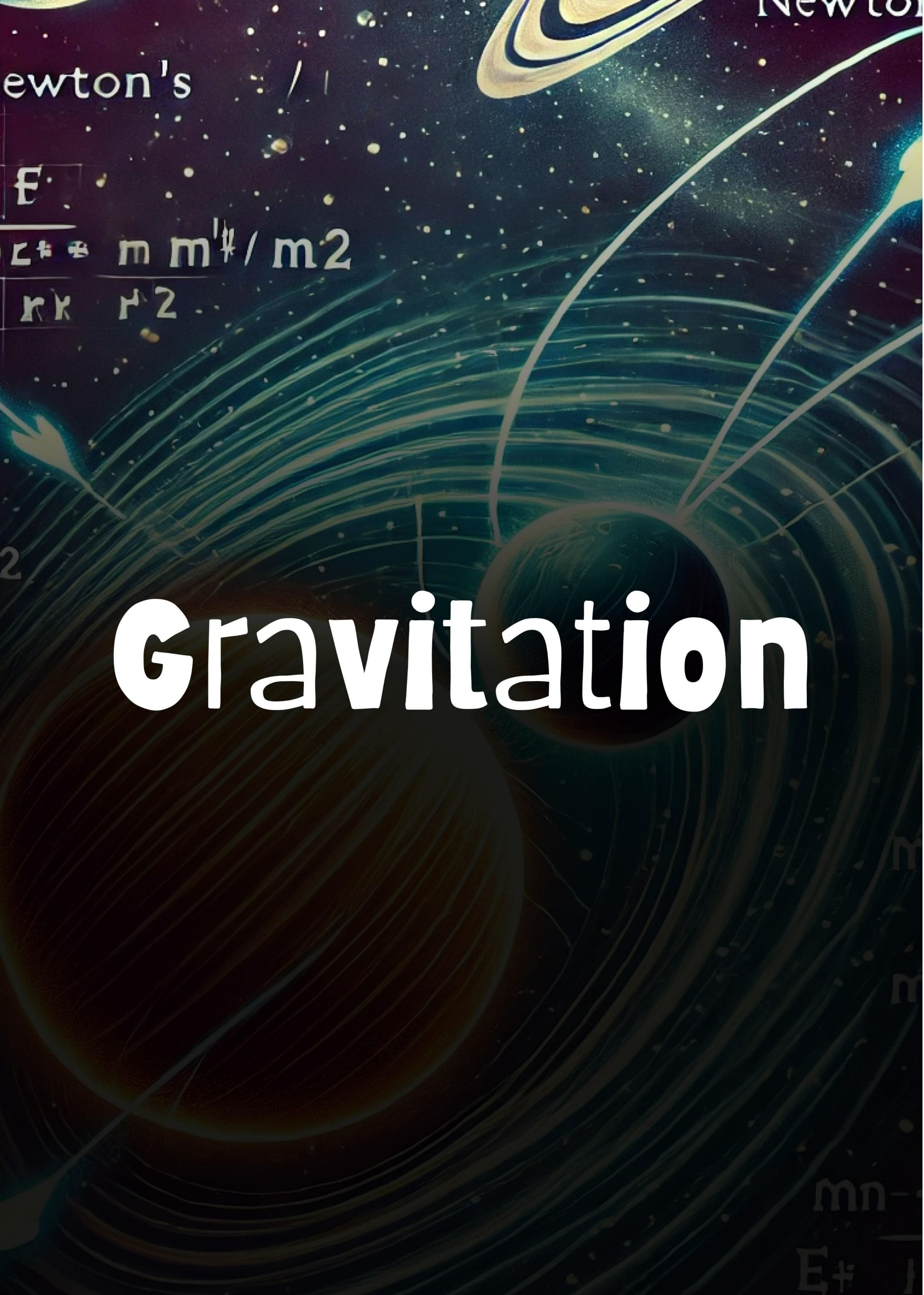


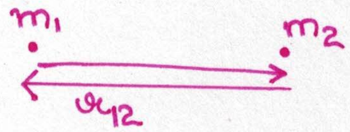
Gravitation

Gravitation

Every particle in the universe attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Mathematically,

$$F_{12} = \frac{G m_1 m_2}{r_{12}^2}$$



where G is the universal gravitation constant.

$$G = 6.67 \times 10^{-11} \ Nm^2/Kg^2$$

SUPERPOSITION PRINCIPLE

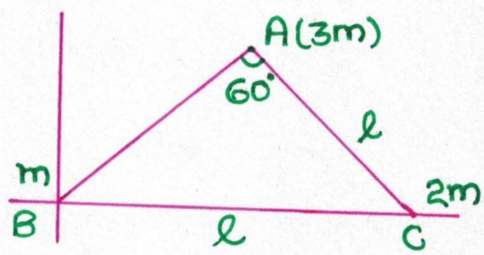
The force of attraction between any particles in a given configuration remains unaffected by presence or absence of any other mass in the universe.

Que.) Find net force on C.

$$\vec{F_c} = \vec{F_{CA}} + \vec{F_{CB}}$$

$$= \frac{6m^2G}{l^2}\left(\frac{\sqrt{3}}{2}(+\hat{j}) - \frac{l}{2}\hat{i}\right)$$

$$= \frac{2m^2G}{l^2} . \hat{i}$$



$$\vec{F_c} = \frac{5m^2G}{l^2}(-\hat{i}) + 3\sqrt{3}\,\frac{Gm^2}{l^2}\,\hat{j}$$

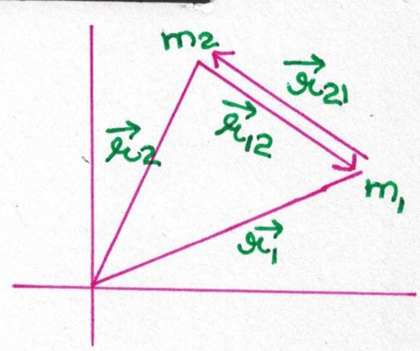
$$= \frac{Gm^2}{l}\left(-5\hat{i} + 3\sqrt{3}\hat{j}\right)$$

VECTOR FORM OF NEWTON'S LAW OF GRAVITATION

$$\vec{F_{21}} = \frac{G m_1 . m_2}{(r_{21})^2}(-\hat{r}_{21})$$

$$\hat{r}_{21} = \frac{\vec{r_{21}}}{r_{21}}$$

$$\boxed{\vec{F_{21}} = \frac{-G m_1 m_2 \vec{r_{21}}}{(r_{21})^3}}$$



GRAVITATION FIELD INTENSITY ($\vec{g}$)

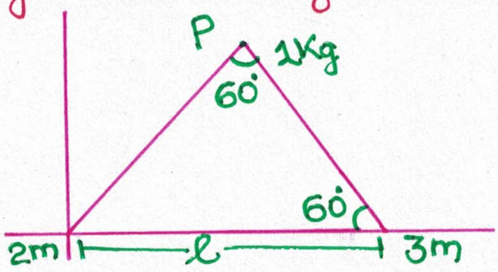
Gravitational force experienced per unit mass at a point in space is called gravitational field intensity.

$$\vec{g} = \frac{\vec{F}}{m}$$

we can see that it is same as acceleration due to gravity.

$$\boxed{\vec{F} = m\vec{g}}$$

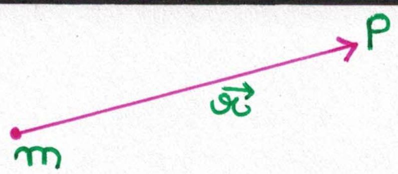
Que.) Find the gravitational field at point P.



$$\vec{F_P} = \frac{5mG}{l^2}\left(-\frac{\sqrt{3}}{2}\hat{j}\right) + \frac{mG}{l^2}\left(\frac{1}{2}\hat{i}\right)$$

$$\vec{g} = \frac{\vec{F_P}}{1\,kg} = \frac{mg\,G}{l^2}\left(-\frac{5\sqrt{3}}{2}\hat{j} + \frac{1}{2}\hat{i}\right)$$

GRAVITATIONAL FIELD DUE TO A POINT MASS



$$\vec{g} = \frac{Gm}{r^2}(-\hat{r})$$

$$\vec{g} = -\frac{Gm}{r^2}\hat{r}$$

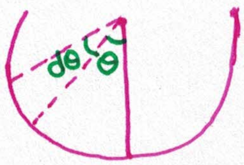
FINDING GRAVITATIONAL FIELD BY INTEGRATION

Since gravitational field is a vector quantity, the components of elemental gravitational field should be integrated separately. Integrating the magnitude of $d\vec{g}$ is meaningless.

$$\vec{g_x} = \int dg_x$$

$$\vec{g_y} = \int dg_y$$

Que.) Find the gravitational field at the center of curvature of a uniform circular wire of mass M and radius R.



$$dM = \frac{M}{\pi}\cdot d\theta$$

$$dg = \frac{G\,dM}{R^2}$$

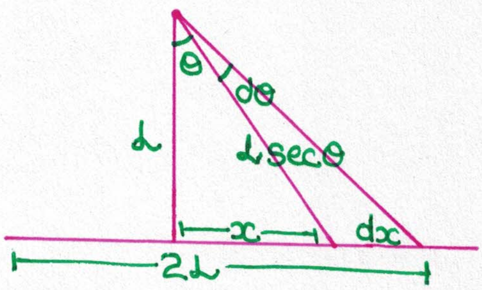
$$dg_y = dg.\cos\theta = \frac{G}{R^2} \times \frac{M}{\pi}\, d\theta.\cos\theta$$

$$g_y = \frac{GM}{\pi R^2}\int_{-\pi/2}^{\pi/2} \cos\theta.d\theta$$

$$\vec{g}_y = \frac{2GM}{\pi R^2} \quad \text{and} \quad \vec{g}_x = 0$$

$$\vec{g} = \frac{2GM}{\pi R^2}$$

Que.) Find the field at the perpendicular bisector at the point P as shown.



$$dM = \frac{M}{2L}\,dx$$

$$dg = \frac{GM}{2L}\,dx \times \frac{1}{L^2 \sec^2\theta}$$

$$d\vec{g}_y = dg.\cos\theta$$

$$= \frac{GM\cos^3\theta}{2L^3}$$

$$x = L\tan\theta$$

$$dx = L\sec^2\theta.d\theta$$

$$dg_y = \frac{GM}{2L^2}\cos\theta.d\theta$$

$$g_y = \int dg_y = \frac{GM}{2L^2}\int_{-\pi/4}^{\pi/4}\cos\theta.d\theta$$

$$= \frac{GM\sqrt{2}}{2L^2}$$

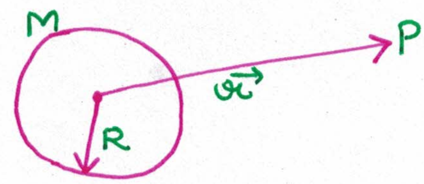
GRAVITATIONAL FIELD DUE TO UNIFORM SPHERICAL SHELL (SHELL THEOREM)

SHELL THEOREM - 1

For an outside point, a uniform spherical shell behaves as if all the mass were concentrated at its own center.

$r > R$

$$\vec{g} = -\frac{GM}{r^2}\hat{r}$$

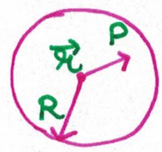


SHELL THEOREM - 2

At an inside point, a shell is as good as it is non-existent.

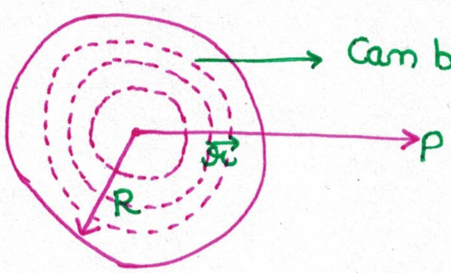
$r < R$

$\vec{g} = 0$



FIELD DUE TO A UNIFORM SOLID SPHERE

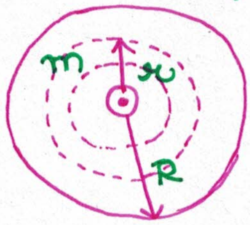
Case 1: For outside point



If $r > R$

$$\vec{g} = -\frac{GM}{r^2}\hat{r}$$

Case 2: For inside point



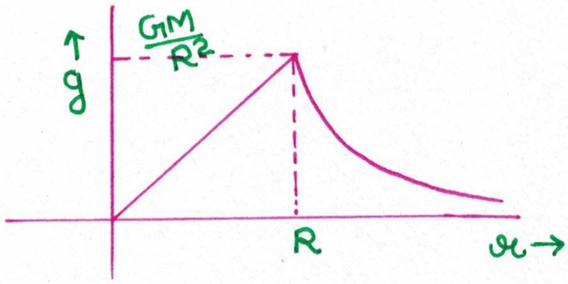
If $r < R$

$$\vec{g} = -\frac{Gm}{r^2}\hat{r}$$

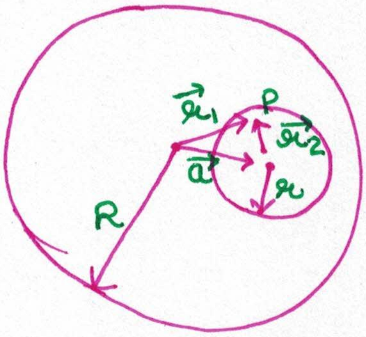
$$m = \frac{M}{\cancel{4/3}\,\cancel{\pi} R^3} \times \cancel{4/3}\,\cancel{\pi} r^3$$

$$m = \frac{Mr^3}{R^3}$$

$$\boxed{\vec{g} = -\frac{GMr}{R^3}\hat{r}}$$



Que.) Show that if a spherical cavity is made in a solid sphere, then the field in the cavity is independent of the point about which it is taken.



$$M = \rho \times \frac{4}{3}\pi R^3$$

$$m = \rho \times \frac{4}{3}\pi r^3$$

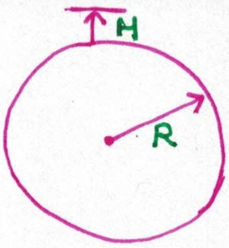
$$\therefore \frac{M}{R^3} = \frac{m}{r^3}$$

$$\vec{g} = -\frac{GM\vec{r}_1}{R^3} - \left(-\frac{Gm\vec{r}_2}{r^3}\right)$$

$$\vec{g} = -\frac{4\pi\rho G}{3}(\vec{r}_1 - \vec{r}_2)$$

$$\vec{g} = -\frac{4\pi}{3}\rho G\vec{a}$$

VARIATION OF ACC^N DUE TO GRAVITY WITH HEIGHT



$$g_{surface} = \frac{GM}{R^2}$$

$$g' = \frac{GM}{(R+H)^2}$$

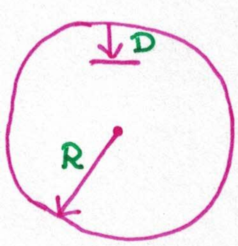
$$\boxed{g' = \frac{GM}{R^2}\left(1+\frac{H}{R}\right)^{-2}}$$

If $H << R$, (max upto 320Km)

$$g' = g\left(1 - \frac{2H}{R}\right)$$

{ Using $(1+x)^n \sim 1+nx$ If $x << 1$ as higher degree terms tend to zero }

VARIATION OF ACC^N DUE TO GRAVITY WITH DEPTH



$$g = \frac{GM}{R^2}$$

$$g' = \frac{GMr}{R^3}$$

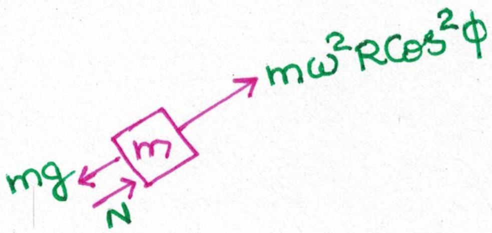
$$g' = g\frac{r}{R}$$

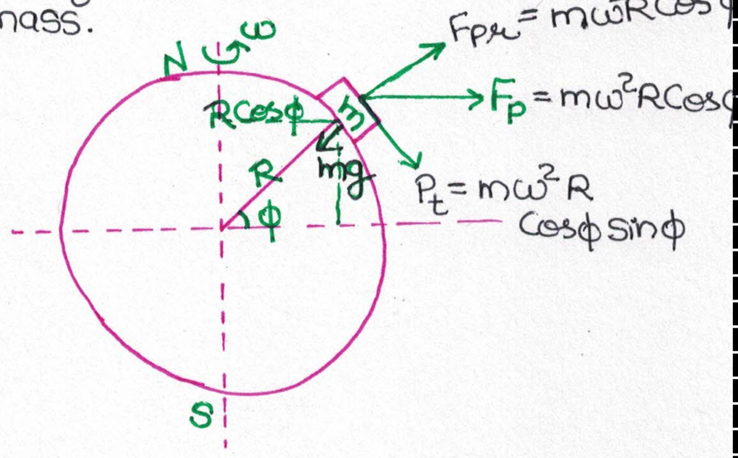
$$g' = g\frac{(R-D)}{R}$$

$$\boxed{g' = g\left(1-\frac{D}{R}\right)}$$

VARIATION OF ACC.^{N} DUE TO GRAVITY WITH LATITUDE

When we talk of variation with latitude, we mean the apparent g due to the effect of centrifugal forces in the direction perpendicular to the surface which is measured as normal reaction per unit mass.





$$N + m\omega^2 R\cos^2\phi = mg$$

$$g' = \frac{N}{m} = g - \omega^2 R\cos^2\phi$$

$$g' = g - \omega^2 R\cos^2\phi$$

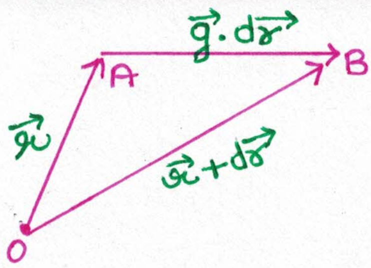
GRAVITATIONAL POTENTIAL (V)

The negative of work done per unit point mass in bringing it from infinity to the given point on space is called gravitational potential at that point of space.

POTENTIAL DIFFERENCE (dV)

Difference in potential is called potential difference.

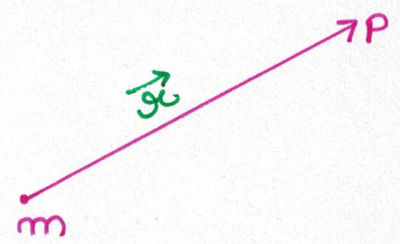
RELATION B/W ELEMENTAL POTENTIAL DIFFERENCE & $\vec{g}$



$$V_B - V_A = dV = -\vec{g}\cdot d\vec{r}$$

by default, we assume potential at ∞ to be 0.

POTENTIAL DUE TO A POINT MASS



$$dv = -\vec{g}\cdot d\vec{r}$$

$$\vec{g} = \frac{-GM}{r^2}\cdot\hat{r}$$

$$d\vec{r} = dr\cdot\hat{r}$$

$$dv = \frac{GM}{r^2}\cdot dr\cdot\hat{r}\cdot\hat{r}$$

$$dv = \frac{GM}{r^2}\cdot dr$$

@ $r=\infty$, $v=0$

@ $r=r$, $v=v$

$$\int_0^v dv = \int_\infty^r \frac{GM}{r^2}\cdot dr$$

$$\boxed{v = \frac{-GM}{r}}$$

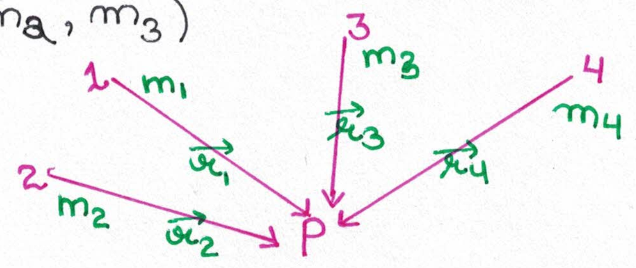
POTENTIAL DUE TO A SYSTEM OF PARTICLE

For a system of particles, potential can again be obtained by superposition.

i.e. $V_P = V_1 + V_2 + V_3 + V_4$

$$V_P = \frac{-GM_1}{r_1} - \frac{GM_2}{r_2} - \frac{GM_3}{r_3}$$

(bringing a unit mass to P from ∞ under influence of m_1, m_2, m_3)

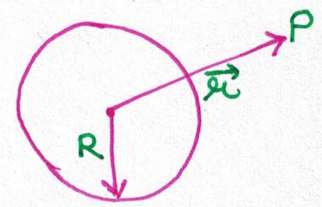


POTENTIAL DUE TO A THIN SPHERICAL UNIFORM SHELL

If $r \geq R$

V is just for point mass

$$v = \frac{-Gm}{r}$$

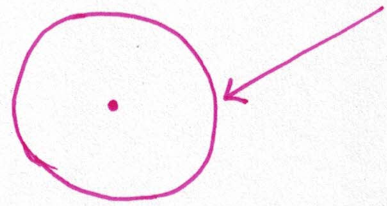


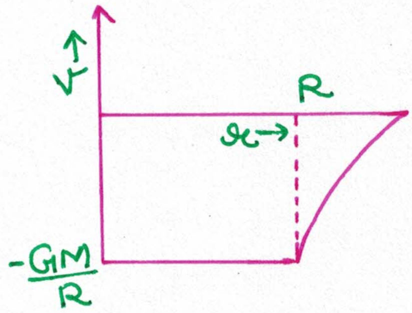
@ $r = R$

$$V = -\frac{GM}{R}$$

If $r < R$,

V is same as that at the surface because inside field is zero.

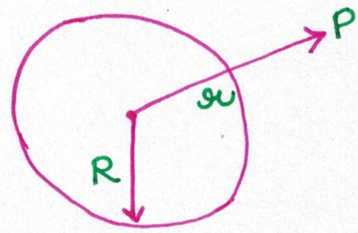




POTENTIAL DUE TO A UNIFORM SOLID SPHERE

If $r \geq R$

V is same as that for a point mass.



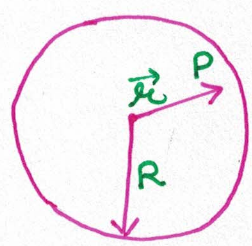
i.e. If $r \geq R$

$$V = -\frac{GM}{r}$$

If $r = R$, $V = V_S$

$$V_S = -\frac{GM}{R}$$

If $r < R$



$$dV = -\vec{g}\cdot d\vec{r}$$

$$\vec{g} = -\frac{GMr}{R^3}\cdot\hat{r}$$

$$d\vec{r} = dr.\hat{r}$$

$$dV = \frac{GMr}{R^3}.dr$$

@ $r = R$, $V = V_3$

@ $r = r$, $V = V_P$

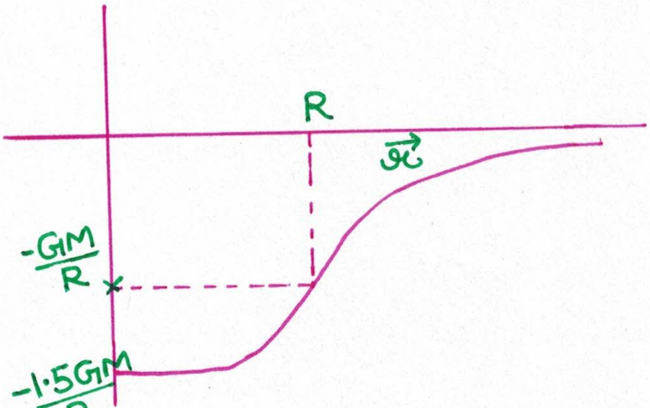
$$\int_{V_S}^{V_P} dV = \frac{GM}{R^3}\int_R^r r.dr$$

$$V_P - V_S = \frac{GM}{R^3}\left[\frac{r^2}{2} - \frac{R^2}{2}\right]$$

$$V_P = V_S + (V_P - V_S)$$

$$V_P = -\frac{GM}{R} + \frac{GM}{R^3}\left[\frac{r^2}{2} - \frac{R^2}{2}\right]$$

$$V_P = -\frac{GM}{R}\left[1.5 - 0.5\frac{r^2}{R^2}\right]$$

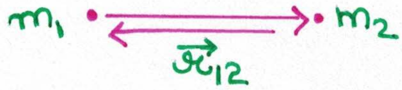


For $r<R$, we will make graph by second derivative of V_P and then for $r>R$, we will check the second derivative of $\left(V = -\frac{GM}{r}\right)$

POTENTIAL ENERGY OF A SYSTEM OF PARTICLE

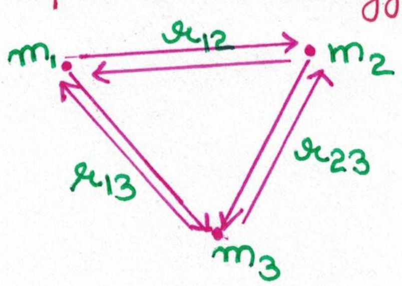
Negative of work done in bringing a set of particles from infinity to the given configuration is called gravitational potential energy of the system of particles.

Que:) Find the P.E. of the shown system of particles.



$$U = -\frac{Gm_1}{r_{12}} \times m_2$$

Que.) Find the potential energy of a system of three particles.



$$-W_1 = 0$$

$$-W_2 = -\frac{Gm_1m_2}{r_{12}}$$

$$V_3 = -\frac{Gm_1}{r_{13}} - \frac{Gm_2}{r_{23}}$$

$$-W_3 = m_3V_3 = -\frac{Gm_2m_3}{r_{23}} - \frac{Gm_1m_3}{r_{13}}$$

$$U = -(W_1 + W_2 + W_3)$$

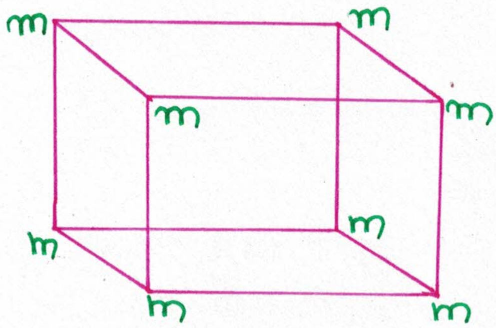
$$U = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_3m_2}{r_{23}}$$

For a system of particles,

$$\boxed{U = -\sum^{*} \frac{Gm_im_j}{r_{ij}}}$$

* r_{13} is same as r_{31} (Don't count twice)

Que.) Find potential energy of the system.



$\{m, m, a\} \rightarrow 12$

$\{m, m, a\sqrt{2}\} \rightarrow 12$

$\{m, m, a\sqrt{3}\} \rightarrow 4$

$$U = -\frac{Gm^2}{a} \times 12 - \frac{Gm^2}{a\sqrt{2}} \times 12 - \frac{Gm^2}{a\sqrt{3}} \times 4$$

Que.) In previous question if mass is removed, then?

$$U = -\frac{Gm^2}{a} \times 9 - \frac{Gm^2}{a\sqrt{2}} \times 9 - \frac{Gm^2}{a\sqrt{3}} \times 3$$

SELF ENERGY OF A BODY

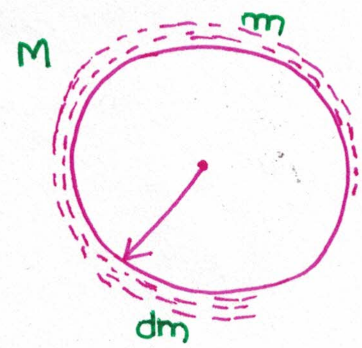
Negative of work done by gravitational forces in assembling a body bringing its infinitesimal elements from infinity is called gravitational self energy.

GRAVITATIONAL SELF ENERGY OF A THIN UNIFORM LAYER

$$dv = -\frac{Gm}{R}.dm$$

$$\int_0^u du = -\int_0^M \frac{Gm}{R}.dm$$

$$u = -\frac{GM^2}{2R}$$

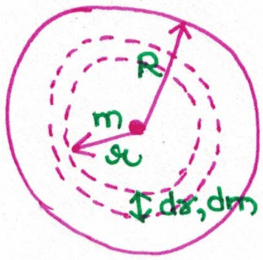


SELF ENERGY OF A SOLID SPHERE

$$du = -\frac{Gm}{r}.dm$$

$$m = \frac{Mr^3}{R^3}$$

$$dm = \frac{M}{R^3}\, 3r^2.dr$$



$$du = \frac{-G}{r}.\frac{Mr^3}{R^3} \times \frac{M}{R^3}.3r^2.dr$$

$$\int_0^u du = -\frac{3GM^2}{R^6}\int_0^R r^4.dr$$

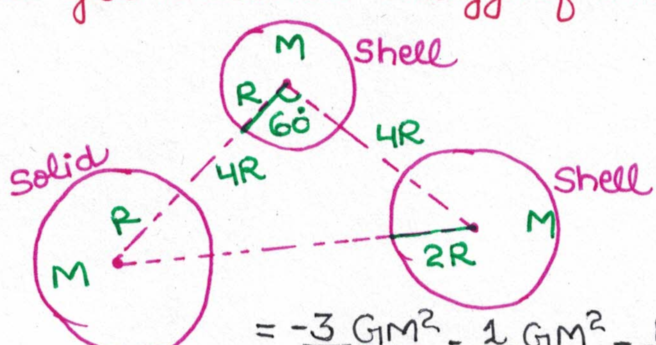
$$\boxed{u = -\frac{3}{5}\frac{GM^2}{R}}$$

TOTAL GRAVITATIONAL ENERGY

Summation of self energy and the interaction potential energy of a system of spheres is called total gravitational potential energy.

Interaction energies of sphere is found by treating them like point masses at their own center.

Que.) Find total gravitational energy of the system.

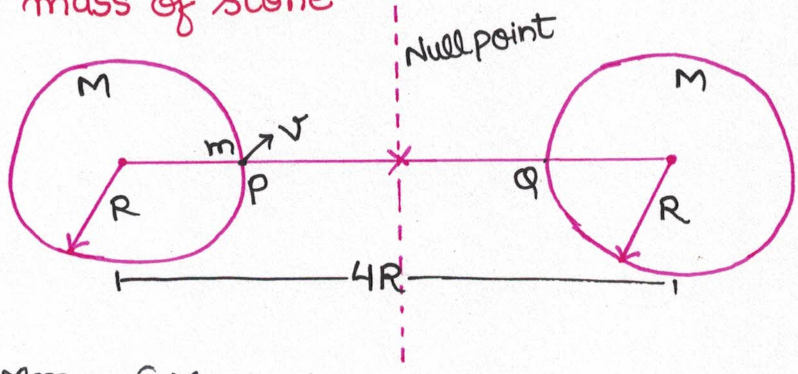


$$= -\frac{3}{5}\frac{GM^2}{R} - \frac{1}{2}\frac{GM^2}{R} - \frac{1}{4}\frac{GM^2}{R} - \frac{GM^2}{4R} - \frac{GM^2}{4R} - \frac{GM^2}{4R}$$

NOTE: We need to write only those energy terms which change in the problem.

Que.) A stone is to be thrown from point P to point Q. Find minimum projection velocity.

$m \rightarrow$ mass of stone



COME

$$-\frac{GMm}{R} - \frac{GMm}{3R} + \frac{1}{2}mv^2 = -\frac{GMm}{\cancel{2}R} \times \cancel{2} + 0$$

$$v = \sqrt{\frac{2GM}{3}}$$

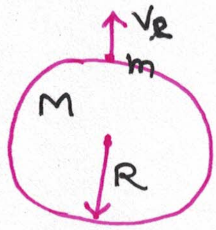
ESCAPE VELOCITY

The minimum speed with which a projectile must be thrown from a heavenly body so as to escape the gravitational field is called the escape velocity.

Que.) Find escape velocity.

$$-\frac{GMm}{R} + \frac{1}{2}mv_e^2 = 0 + 0$$

limiting case
K.E. @ ∞



$$v_e = \sqrt{\frac{2GM}{R}} \approx \sqrt{2gR} = 11.2 \text{ Km/sec.}$$

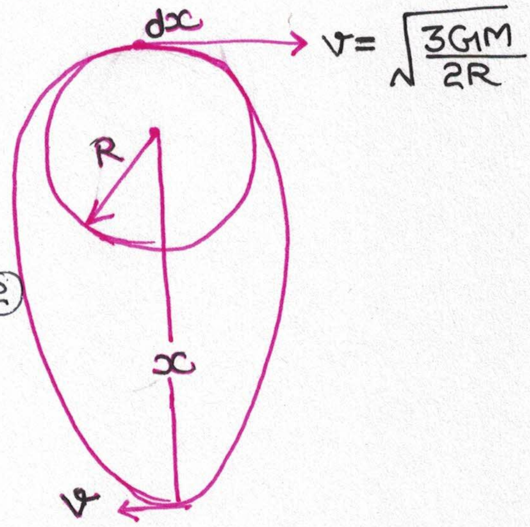
Problems Involving projection of small projectiles from heavenly bodies.

In such problems we can typically make the following equations:

(1) CAM about centre of heavenly body (torque of gravitational force about centre of heavenly body.

(2) COME (neglecting air drag)

Que.) Calculate the farthest distance of the projectile from the centre of the earth.



CAM

$$m\sqrt{\frac{3GM}{2R}}\cdot R = mvx \quad -(1)$$

COME

$$-\frac{GMm}{R}+\frac{1}{2}m\cdot\frac{3GM}{2R} = -\frac{GMm}{x}+\frac{1}{2}mv^2 \quad -(2)$$

$$v^2 = \frac{3GMR}{2x^2}$$

$$v^2 = 2GM\left[\frac{1}{x}-\frac{1}{4R}\right]$$

$$\frac{3R}{2x^2} = \frac{2}{x}-\frac{1}{2R}$$

$$3R^2 = 4xR - x^2$$

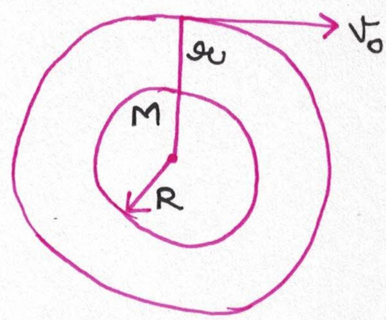
$$x^2 - 4xR + 3R^2 = 0$$

$$(x-3R)(x-R) = 0$$

$$\therefore x = 3R \quad \text{or} \quad x = R$$

ORBITAL VELOCITY

The speed with which a projectile must be launched perpendicular to a radial line so as to execute a circular orbit is called orbital velocity.



$$\frac{GMm}{r^2} = \frac{mv_o^2}{r^2}$$

$$v_o = \sqrt{\frac{GM}{r}}$$

For near surface, $r = R$

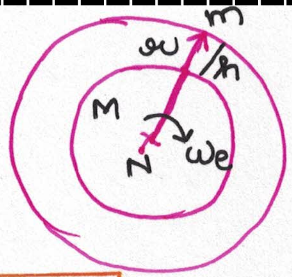
$$v_o = \sqrt{\frac{GM}{R}} = \frac{v_e}{\sqrt{2}} = \sqrt{gR}$$

GEO-STATIONARY SATELLITE

A satellite in the equitorial plane which appear to be stationary with respect to a point on earth is called geo-stationary satellite.

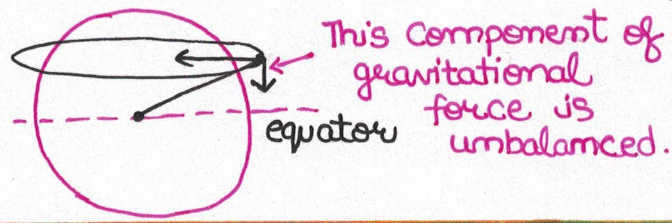
$$\frac{GMm}{r^2} = m\omega_e^2 r$$

$$r = \left(\frac{GM}{\omega_e^2}\right)^{1/3}$$



$h = 36000$ Km above surface of earth.

NOTE: It is always in equatorial plane. Why?



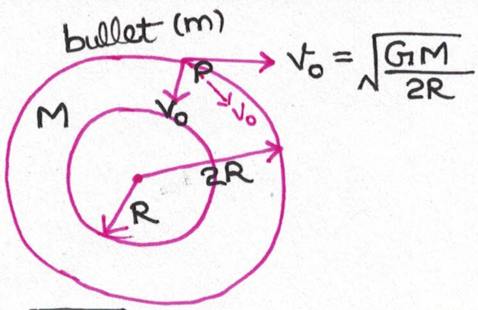
POLAR SATELLITE

A satellite whose orbit passes above the north and south pole is called as polar satellite.

Such satellites are used for scanning the surface of earth (indirectly spying, remote sensing) and climate change.

As its time of revolution and that of earth has no L.C.M. (Irrational no.s don't have L.C.M.).

Que.) A bullet is thrown from a satellite P with a velocity v_0 with respect to it. Will the bullet touch the earth's surface.



CAM $\quad mv x = m\sqrt{\frac{GM}{2R}} \times 2R \quad$ —①

COME $\quad -\frac{GMm}{2R} + \frac{1}{2}m\left(\left(\frac{GM}{2R}\right)+\left(\frac{GM}{2R}\right)\right) = -\frac{GMm}{x} + \frac{1}{2}mv^2$ —②

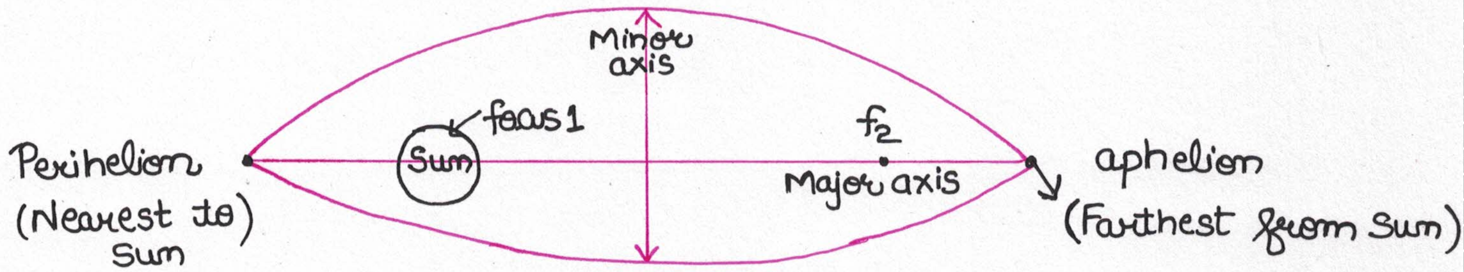
① $\Rightarrow v^2 = \frac{2GMR}{x^2}$

② $\Rightarrow v^2 = \frac{2GM}{x}$

$\Rightarrow \frac{2G\cancel{M}R}{x^{\cancel{2}}} = \frac{2G\cancel{M}}{\cancel{x}}$

$\Rightarrow \quad x = R$

PLANETARY ORBITS NOMENCLATURE



- ★ Perigee and apogee are used when satellite is revolving around earth.
- ★ Perihelion and aphelion are used when earth is revolving around Sun.
- ★ P.E. is maximum at aphelion $\left(-\frac{GMm}{r}\right)$ & K.E. is minimum.
- ★ Speed is perpendicular to the distance at perihelion and aphelion.
- ★ Sum of distance from the two foci remains constant.
- ★ K.E. is maximum at perihelion and P.E. is minimum.

KEPLER'S LAWS

FIRST LAW

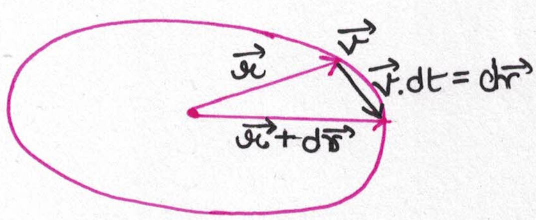
All the planets revolve around Sun in elleptical orbitals, nearly circular.

SECOND LAW

The aerial velocity of planets around the Sun is constant (area swept per unit time by the line joining the Sun to the planet).

PROOF OF SECOND LAW

$$dA = \left|\frac{1}{2}\times\vec{r}\times\vec{v}\,dt\right|$$



$$\frac{dA}{dt} = \frac{1}{2}\left|\vec{r}\times\vec{v}\right|$$

$$\frac{dA}{dt} = \frac{1}{2}\left|\vec{r}\times m\vec{v}\right|\times\frac{1}{m}$$

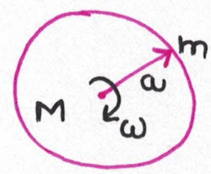
$$\frac{dA}{dt} = \frac{|\vec{L}|}{2m}$$

Since $|\vec{L}|$ = Constant, aerial velocity (dA/dt) for a particular planet is constant.

THIRD LAW

The square of time period of a revolution of a planet is proportional to the cube of semi-major axis of the planet.

PROOF OF THIRD LAW



$$\cancel{m}\omega^2 a = \frac{GM\cancel{m}}{a^2}$$

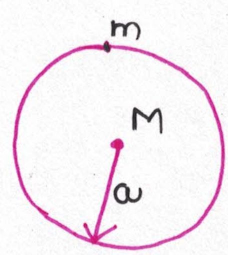
$$T = \frac{2\pi}{\omega} \quad , \quad \omega = \frac{2\pi}{T}$$

$$\frac{4\pi^2}{T^2} = \frac{GM}{a^3}$$

$$T^2 = \frac{4\pi^2}{GM} \cdot a^3$$

NOTE: If we take M as Mass of earth then T is of satellite and if M is mass of Sun then T is of planet, and this expression holds good for elleptical orbits also.

TOTAL ENERGY OF A PLANET IN AN ORBIT



$$v_0 = \sqrt{\frac{GM}{a}}$$

$$K.E. = \frac{1}{2} m v_0^2$$

$$K.E. = \frac{GMm}{2a}$$

$$P.E. = -\frac{GMm}{a}$$

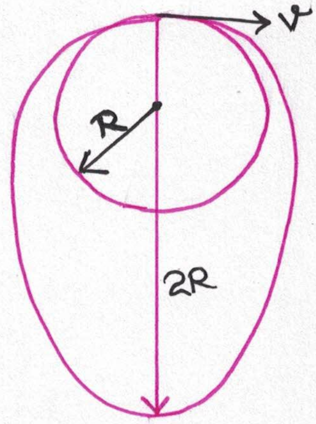
$$T.E. = K.E. + P.E.$$

$$= +\frac{GMm}{2a} - \frac{GMm}{a}$$

$$\boxed{T.E. = -\frac{GMm}{2a}}$$

NOTE: The same result holds for elleptical orbits also.

Que.) What should be the value of v so that maximum distance of the satellite from centre is 2R?

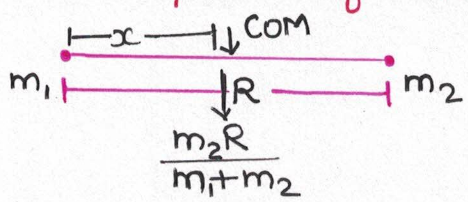


$$\frac{1}{2}mv^2 - \frac{GMm}{R} = -\frac{GMm}{2\times\frac{3}{2}R}$$

$$\frac{1}{2}mv^2 = \frac{2GMm}{3R}$$

$$v = 2\sqrt{\frac{GM}{3R}}$$

Que.) Find the time period of revolution of binary star.



$$\frac{Gm_1m_2}{R^2} = m_1\omega^2\frac{m_2R}{m_1+m_2}$$

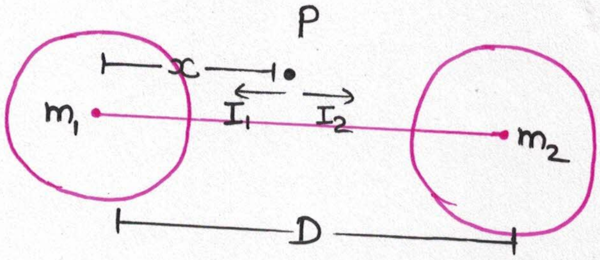
$$\omega^2 = \sqrt{\frac{G(m_1+m_2)}{R^3}}$$

$$T = \frac{2\pi}{\omega}$$

POINT OF ZERO INTENSITY

x = point at which the force of attraction b/w the two bodies is zero.

$$x = \frac{\sqrt{m_1}}{\sqrt{m_1}+\sqrt{m_2}}D$$



Proof: $\vec{I_1} + \vec{I_2} = 0$

$$-\frac{Gm_1}{x^2} + \frac{Gm_2}{(d-x)^2} = 0$$

$$\frac{m_1}{x^2} = \frac{m_2}{(d-x)^2}$$

$$m_1(d^2+x^2-2dx) = m_2x^2$$

$$(m_1-m_2)x^2 - 2m_1dx + m_1d^2 = 0$$

$$x = \frac{2m_1d \pm \sqrt{4m_1^2d^2 - 4(m_1-m_2)(m_1d^2)}}{2(m_1-m_2)}$$

$$x = \frac{m_1d \pm \sqrt{m_1m_2d}}{(m_1-m_2)}$$